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A STOCHASTIC APPROACH OF THE ENERGY ANALYSIS FOR ONE-DIMENSIONAL STRUCTURES

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This paper presents a complete and rigourous derivation of the well-known power flow equations, by introducing two types of Gaussian random parameters in the description of the studied structures: the first one deals with the spatial position, and the latter with the location of the boundaries. Investigations are carried out for the case of one-dimensional systems (longitudinal vibrations in rods and transverse displacements in beams). The Simplified Energy Method equations are found to be the asymptotic form of the random relationships, when the frequency as well as the standard deviation are sufficiently high. Moreover, the input powers used in SEM models are restored by the stochastic formulation.

1. INTRODUCTION

The prediction of high frequency phenomena is an up-to-date subject for engineers dealing with vibro acoustic and noise transmission in built-up structures. One of the most commonly used formulation to carry out these predictions, is the Statistical Energy Analysis (SEA) [1]. An alternative to the SEA is the Power Flow Analysis based on the original work by Belov *et al*. [2]. Later , Nefske and Sung [3] applied the finite element method to deal with the Power Flow Analysis in a straight transversally vibrating beam. Wohlever, Bouthier and Bernhard [4–6] gave further results concerning the energy models of rods, Euler–Bernoulli beams, membranes and Kirchhoff–Love plates. More recently, Lase *et al*. [7] elaborated the so-called General Energy Method (GEM), which provides a complete energy description of rods and beams, using the reactive power flow, the Lagrangian energy density, the active power flow and the total energy density. Several high frequency assumptions were then applied by the authors to the GEM equations. The final simplified form of the GEM is called the Simplified Energy Method (SEM).

The aim of this paper is to reobtain the SEM and Power Flow Analysis predictions for rods and beams, by introducing random parameters in the description of the geometrical parameters of the structures. The stochastic modelling illustrates the unvoidable degree of uncertainty linked to the high frequency vibrational response [8]. The expectations of the energy variables are

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evaluated, and it is shown that the SEM energy governing equations are the limit of the stochastic formulation results when the standard deviation of the random law increases.

2. PREAMBLE

The present study is concerned with Power Flow Analysis, carried out under the classical assumptions of elastodynamics, harmonic and steady state conditions. In this context, the basic energy variables required to describe the vibrational behaviour of rods and beams are the instantaneous kinetic energy density $T(M, t)$, the strain energy density $U(M, t)$, and the active energy flow $I(M, t)$. The general expressions of these variables are:

$$\text{for the rod} \begin{cases} T(M, t) = (1/2)\rho S \,\mathrm{Re}\left\{\dfrac{\partial u(t)}{\partial t}\right\}^2, \\[2ex] U(M, t) = (1/2)\,\mathrm{Re}\left\{ES\dfrac{\partial u(t)}{\partial x}\right\}^2, \\[2ex] I(M, t) = -\mathrm{Re}\left\{ES\dfrac{\partial u(t)}{\partial x}\right\}\mathrm{Re}\left\{\dfrac{\partial u(t)}{\partial t}\right\}, \end{cases} \tag{1}$$

$$\text{for the beam} \begin{cases} T(M, t) = (1/2)\rho S \,\mathrm{Re}\left\{\dfrac{\partial v(t)}{\partial t}\right\}^2, \\[2ex] U(M, t) = (1/2)\,\mathrm{Re}\left\{EI\dfrac{\partial^2 v(t)}{\partial x^2}\right\}^2, \\[2ex] I(M, t) = -\mathrm{Re}\left\{EI\dfrac{\partial^3 v(t)}{\partial x^3}\right\}\mathrm{Re}\left\{\dfrac{\partial v(t)}{\partial t}\right\}, \\[2ex] \qquad + \mathrm{Re}\left\{EI\dfrac{\partial^2 v(t)}{\partial x^2}\right\}\mathrm{Re}\left\{\dfrac{\partial^2 v(t)}{\partial x\,\partial t}\right\}. \end{cases} \tag{2}$$

$\mathrm{Re}\{u(t)\}$ denotes the longitudinal displacement whereas, $\mathrm{Re}\{v(t)\}$ is the transversal displacement. The parameter ρ denotes the mass density, S is the cross section, $EI = E_0(1 + \mathrm{i}\eta)I$ is the flexural rigidity. Using the expressions of $T(M, t)$ and $U(M, t)$, one can define the instantaneous energy density $W(M, t)$ and lagrangian density $L(M, t)$:

$$W(M, t) = T(M, t) + U(M, t), \qquad L(M, t) = T(M, t) - U(M, t). \tag{3}$$

In the following investigations the energy variables are time averaged harmonic functions. In this context, one can use the property of the produce of the real and imaginary part of complex harmonic functions:

$$\langle \mathrm{Re}\,(f)\,\mathrm{Re}\,(g)\rangle_t \overset{\mathrm{def}}{=} \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{Re}\,(f)\cdot \mathrm{Re}\,(g)\,\mathrm{d}t = \tfrac{1}{2}\,\mathrm{Re}\,(f\cdot g^*), \tag{4}$$

where f and g represent complex functions. Therefore, the time averages of the different energy variables are:

$$\text{for the rod}\begin{cases} \langle T(M,t)\rangle_t = (1/4)\rho S\omega^2 uu^*, \\ \langle U(M,t)\rangle_t = (1/4)E_0 S\dfrac{\partial u}{\partial x}\dfrac{\partial u^*}{\partial x}, \\ \langle I(M,t)\rangle_t = -\mathrm{Re}\,\{(\mathrm{i}\omega/2)ES(\partial u/\partial x)u^*\}, \end{cases} \tag{5}$$

$$\text{for the beam}\begin{cases} \langle T(M,t)\rangle_t = (1/4)\rho S\omega^2 vv^*, \\ \langle U(M,t)\rangle_t = (1/4)E_0 I\dfrac{\partial^2 v}{\partial x^2}\dfrac{\partial^2 v^*}{\partial x^2}, \\ \langle I(M,t)\rangle_t = -\mathrm{Re}\left\{(\mathrm{i}\omega/2)EI\left\{\dfrac{\partial^3 v}{\partial x^3}v^* - \dfrac{\partial^2 v}{\partial x^2}\dfrac{\partial v^*}{\partial x}\right\}\right\}, \end{cases} \tag{6}$$

where u and v are the respective complex displacements associated to the variables $u(t) = u\exp(\mathrm{i}\omega t)$ and $v(t) = v\exp(\mathrm{i}\omega t)$.

In the following parts, the time averaged energy variables will be used exclusively. Dealing with Energy Flow Analysis requires the use of the basic energy balance equation. In steady state conditions and harmonic loading, this equation is written [7]:

$$\frac{\partial\langle I(M,t)\rangle_t}{\partial x} + \eta\omega(\langle W(M,t)\rangle_t - \langle L(M,t)\rangle_t) = \langle p_0(x_f,t)\rangle_t, \tag{7}$$

where $\langle p_0(x_f,t)\rangle_t$ is the time averaged input power and x_f is the location of the loading.

3. THE SIMPLIFIED ENERGY FLOW ANALYSIS FOR ONE-DIMENSIONAL STRUCTURES

High frequency energy flow analysis for longitudinally vibrating rods and transversally vibrating beams has been carried out by different research groups. Similar energetic relationships have been proposed, even if the underlying phenomenological principles differ from one author to another.

Lase *et al*. [7] developed the GEM, which consists of expliciting the different terms of the harmonic complex energy flow balance, using a wave description. This formulation leads to two differential equations involving respectively the total energy density, and the active power flow, and the Lagrangian density and the reactive power flow. Two simplifications of the GEM are proposed by Ichchou *et al*. [9]. The evanescent wave field is neglected far from the loading areas and the discontinuities of the structure, and the interferences between the propagative waves is not taken for granted. These assumptions lead to neglecting the Lagrangian density and reactive power flow. The new equations obtained after the simplifications are the basis of the SEM.

The starting point of Wohlever and Bernhard [4] is quite different. The Power Flow Analysis is derived from the real part of the harmonic energy flow balance. The near field contribution is neglected and a spatial average over a wavelength of the different terms of the latter equation is considered. A relationship is directly obtained between the explicit spatial averages of the total energy density and the active power flow.

The main result of the developments of Lase and Wohlever stands in the following equation, valid for both rods and beams:

$$\langle\langle I\rangle_t\rangle = -\frac{c_g^2}{\eta\omega}\frac{\partial\langle\langle W\rangle_t\rangle}{\partial x}, \tag{8}$$

where c_g represents the group velocity. $\langle\langle\ \rangle_t\rangle$ denotes the time average plus space average over a half wavelength according to Wohlever, whereas Ichchou *et al*. [9] defines this symbol as the time average of the energy variables for which the interferences between the propagative waves are neglected (as explained previously). Using equation (8) and a simplified form of the dissipative relationship [10], Wohlever and Lase finally obtained the following expression of the energy balance equation, for a non-loaded region:

$$\frac{\partial^2\langle\langle W\rangle_t\rangle}{\partial x^2} - \frac{\eta^2 c_g^2}{\omega^2}\langle\langle W\rangle_t\rangle = 0. \tag{9}$$

Equation (9) is solved using energy conditions, in terms of input power and boundary relationships. The exact input power (difficult to evaluate) is commonly replaced by the power of the associated infinite or semi-infinite structure [7].

The goal of the following sections is to reobtain these relationships by introducing random parameters in the definition of the studied structures. In this context, the expectations of the energy variables are evaluated, and compared to the results obtained by the SEM.

4. THE RANDOM DESCRIPTION

4.1. A GENERAL OVERVIEW

In this section, a random approach of the high frequency field is introduced. The general idea is that any attempt to deterministic high frequency prediction is unrealistic. The reason is that the modal response of mechanical systems is more and more sensitive to small perturbations of the geometrical and mechanical parameters of the structures, when the frequency increases [8]. Manohar and Keane [11] highlight this phenomenon by calculating the successive density functions of the eigenfrequencies of a beam, for which a random parameter is introduced in the definition of its mass density.

There are numerous possibilities concerning the choice of structural parameters on which can be applied the random variables. Actually, the randomness of the structure studied in the high frequency field is resulting from small errors occuring on the global description of the system. Therefore, a stochastic process could be introduced on the mechanical parameters, such as the mass density or the modulus of elasticity. In the following analytical formulations, the random parameters are introduced in the geometrical description of the structures: the boundary co-ordinates (x_1 and x_2), the loading location (x_f) and the spatial position (x). These different geometrical random variables are considered independent two by two. According to Fahy and Mohammed [12], the choice of geometrical randomness is relevant. Indeed, when considering individual members of sets of physical systems which share the same gross characteristics (such as cars leaving a production line), they differ in geometrical details which can significantly influence the vibrational behaviour. The following notation is introduced:

$$\begin{cases} \tilde{x}_1 = x_1 + \varepsilon_1, \\ \tilde{x}_2 = x_2 + \varepsilon_2, \\ \tilde{x}_f = x_f + \varepsilon_f, \\ \tilde{x} = x + \varepsilon, \end{cases} \tag{10}$$

where ε_1, ε_2, ε_f, and ε are Gaussian random variables of zero mean and respective standard deviations σ_1, σ_2, σ_f and σ_x. x_1, x_2, x_f and x denote the means of $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_f$ and $\tilde{x}$, respectively.

The energy random variables of the rod and the beam may be expressed using the different definitions set up previously. Different relationships between the expectations of the energy variables will be exhibited.

The expectation calculation process of a function (f) of n random independent Gaussian variables ε_i ($i = 1, n$), whose respective means and standard deviations are m_i and σ_i, is recalled:

$$\langle f(\varepsilon_1,\ldots,\varepsilon_n)\rangle_{\varepsilon_1,\ldots,\varepsilon_n} = \frac{1}{\sqrt{2\pi}^{\,n}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(y_1, \ldots, y_n) \prod_{i=1}^{n} \frac{\mathrm{e}^{-(y_i - m_i)^2/2\sigma_i^2}}{\sigma_i} \, \mathrm{d}y_1 \ldots \mathrm{d}y_n \tag{11}$$

where $\langle - \rangle_{\varepsilon_1,\ldots,\varepsilon_n}$ denotes the expectation with respect to $\varepsilon_1, \ldots, \varepsilon_n$.

4.2. THE ENERGY RANDOM FORMULATION OF THE ROD

The expression of the longitudinal displacement of a rod submitted to a point harmonic loading, is a solution of the classical equation of motion [13], whose general solution is expressed in terms of superposition of two propagating waves:

$$u(x) = A_1(x_1, x_2) \exp(-\mathrm{i}kx) + A_2(x_1, x_2) \exp(\mathrm{i}kx), \tag{12}$$

where u denotes the longitudinal displacement of the forced problem provided that a subdomain not containing x_f is considered. The values of A_1 and A_2 represent the amplitudes of the travelling waves, they are obtained utilizing the boundary conditions. $E = E_0(1 + \mathrm{i}\eta)$ is the complex modulus and k denotes the wave number defined by,

$$k = \sqrt{\omega^2\rho/E_0(1 + \mathrm{i}\eta)} \approx k_0(1 - \mathrm{i}\eta/2). \tag{13}$$

The parameter η denotes the hysteretic damping factor, describing the dissipative characteristic of the rod, while x_1 and x_2 respectively denote the co-ordinates of the boundaries of the rod. The time averaged kinetic energy density, the strain energy density and the active power flow, may be written as:

$$\begin{cases} \langle T(x)\rangle_t = \dfrac{\rho S\omega^2}{4} uu^*, \\ \langle U(x)\rangle_t = \dfrac{E_0 S}{4}\dfrac{\partial u}{\partial x}\dfrac{\partial u^*}{\partial x}, \\ \langle I(x)\rangle_t = \mathrm{Re}\left\{\dfrac{-\mathrm{i}ES\omega}{2}\dfrac{\partial u}{\partial x}u^*\right\}. \end{cases} \tag{14}$$

The geometrical random parameters are introduced in the expression of the energy variables (14). The expressions of the time averaged energy density, Lagrangian density and active power flow may then be obtained using equations (3), (12) and (14). The complete developments are given by Bouthier [5] and Lase *et al*. [7] for the particular case of the rod:

$$\begin{cases} \langle W(\tilde{x})\rangle_t = \frac{1}{2}\rho S\omega^2\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0\tilde{x}} + |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0\tilde{x}}\}, \\ \langle L(\tilde{x})\rangle_t = \frac{1}{2}\rho S\omega^2\{A_1(\tilde{x}_1, \tilde{x}_2)A_2^*(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{-2\mathrm{i}k_0\tilde{x}} + A_1^*(\tilde{x}_1, \tilde{x}_2)A_2(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{2\mathrm{i}k_0\tilde{x}}\}, \\ \langle I(\tilde{x})\rangle_t = \frac{1}{2}\rho Sc_g\omega^2\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0\tilde{x}} - |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0\tilde{x}}\}, \end{cases} \tag{15}$$

where $c_g = \omega/k_0$ is the group velocity. The expectations with respect to $\tilde{x}$ of the energy variables given by the relationships (15) are then evaluated. The details of calculations of these expectations are given in the Appendix. One finally obtains:

$$\begin{cases} \langle\langle W(\tilde{x})\rangle_t\rangle_x = \frac{1}{2}\rho S\omega^2\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2k_0^2\sigma_x^2/2} + |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2k_0^2\sigma_x^2/2}\}, \\ \langle L(\tilde{x})\rangle_t = \frac{1}{2}\rho S\omega^2\{A_1(\tilde{x}_1, \tilde{x}_2)A_2^*(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{-2\mathrm{i}k_0 x}\,\mathrm{e}^{-k_0^2\sigma_x^2/2} \\ \qquad + A_1^*(\tilde{x}_1, \tilde{x}_2)A_2(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{2\mathrm{i}k_0 x}\,\mathrm{e}^{-k_0^2\sigma_x^2/2}\}, \\ \langle I(\tilde{x})\rangle_t = \frac{1}{2}\rho Sc_g\omega^2\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2k_0^2\sigma_x^2/2} - |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2k_0^2\sigma_x^2/2}\}, \end{cases} \tag{16}$$

where $\langle - \rangle_x$ represents the expectation with respect to the random variable $\tilde{x}$. Using the relationships (16), one can easily prove that the energy density and the active power flow satisfy the classical relationship (8):

$$\langle\langle I(\tilde{x})\rangle_t\rangle_x = \frac{c_g^2}{\eta\omega}\frac{\partial\langle\langle W(\tilde{x})\rangle_t\rangle_x}{\partial x}. \tag{17}$$

The study of $\langle\langle L(\tilde{x})\rangle_t\rangle_x$ leads to the conclusion that this energy variable vanishes quickly to zero. Indeed, the term $\exp[-2k_0^2\sigma_x^2]$ decreases very rapidly when $k_0\sigma_x$ grows. It is then possible to introduce the relationship (17) in the energy balance equation (7). Furthermore, considering the expectation of the Lagrangian density equal to zero leads to the following expression of the expectation with respect to $\tilde{x}$ of the energy balance equation:

$$-\frac{c_g^2}{\eta\omega}\frac{\partial^2\langle\langle W(\tilde{x})\rangle_t\rangle_x}{\partial x^2}+\eta\omega\langle\langle W(\tilde{x})\rangle_t\rangle_x=0. \tag{18}$$

The injected power does not appear in the equation since the relationship (18) is written on a subdomain not submitted to an external loading. Equation (18) is the fundamental relationship governing the expectation of the time averaged energy density when the term $\exp[-2k_0^2\sigma_x^2]$ is small enough to allow the simplifications proposed before. This equation is similar to the one obtained by Lase *et al*. [7] and Bouthier [5]. In order to solve this equation, boundary conditions are required. These conditions are expressed in terms of Neumann, Dirichlet or Cauchy boundary conditions, given on the energy density variable or its first derivative, corresponding to the expectation of the active power flow, according to equation (17). The boundary conditions for usual cases of rods are given by Lase *et al*. [7]. Moreoever, the injected power is introduced at a boundary of a non-loaded subdomain, on which the energy equation is valid. Therefore, the injected power must be evaluated for x_f equal to x_1 or x_2. For complex structures, the exact value of the input power is unavailable, and generally one replaces it by the input power associated to the impedance of a semi-infinite rod [3], whose expression is:

$$p_{inf}=\tfrac{1}{2}|F_0|^2\,\mathrm{Re}\,\{1/\rho S c_g\}. \tag{19}$$

In the remainder of this section, it will be shown that one can approximate $\langle\langle p_0(\tilde{x}_1,\tilde{x}_2)\rangle_t\rangle_{x_1,x_2}$ by p_{inf} in the high frequency field. The symbol $\langle-\rangle_{x_1,x_2}$ denotes the expectations with respect to $\tilde{x}_1$ and $\tilde{x}_2$. Therefore, the following energy balance equation is finally considered, in order to express the input power condition in terms of p_{inf}:

$$-\frac{c_g^2}{\eta\omega}\frac{\partial^2\langle\langle\langle W(\tilde{x})\rangle_t\rangle_x\rangle_{x_1,x_2}}{\partial x^2}+\eta\omega\langle\langle\langle W(\tilde{x})\rangle_t\rangle_x\rangle_{x_1,x_2}=0. \tag{20}$$

The particular case of the cantilever rod loaded at $x_f=x_1$ (shown in Figure 1) for which the explicit expression for $\langle p_0(\tilde{x}_1,\tilde{x}_2)\rangle_t$ is reachable, is now considered. In

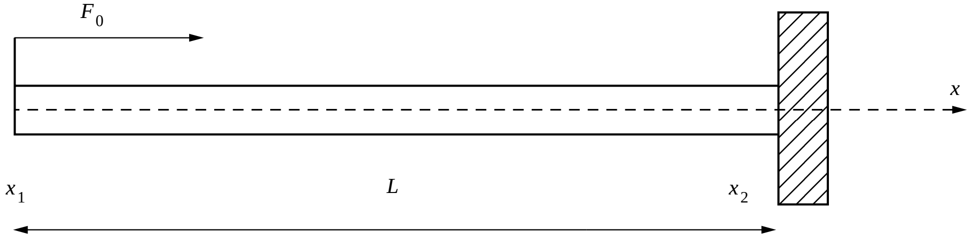

Figure 1. Cantilever rod submitted to a point loading at the free end.

the following developments, it is shown that different conditions are required to use p_{inf} in the stochastic formulation.
The expression of the time averaged power input is:

$$\langle p_0(x_1, x_2)\rangle_t = -\mathrm{Re}\left\{\frac{\mathrm{i}\omega}{2} F_0 u^*(x_1)\right\}, \tag{21}$$

and the expression of the displacement for a cantilever rod loaded at its free end is:

$$u(x) = \frac{F_0}{2\mathrm{i}kES}\frac{\mathrm{e}^{2\mathrm{i}k(x_2-x_1)}\,\mathrm{e}^{-\mathrm{i}k(x-x_1)} - \mathrm{e}^{\mathrm{i}k(x-x_1)}}{1+\mathrm{e}^{2\mathrm{i}k(x_2-x_1)}}. \tag{22}$$

According to equations (21) and (22), and by replacing x_1 and x_2 by their associated random variables $\tilde{x}_1$ and $\tilde{x}_2$, one can write the expression of the random input power as follows:

$$\langle p_0(\tilde{x}_1, \tilde{x}_2)\rangle_t = \mathrm{Re}\left\{\frac{|F_0|^2}{2\rho S c_g}\frac{\mathrm{e}^{2\mathrm{i}k_0(1+\mathrm{i}\eta/2)(\tilde{x}_2-\tilde{x}_1)} - 1}{\mathrm{e}^{2\mathrm{i}k_0(1+\mathrm{i}\eta/2)(\tilde{x}_2-\tilde{x}_1)} + 1}\right\}. \tag{23}$$

The input power may be rewritten in terms of a Taylor series expansion with respect to the term $\exp[2\mathrm{i}k_0(\tilde{x}_2-\tilde{x}_1) - \eta k_0(\tilde{x}_2-\tilde{x}_1)]$ whose modulus is lower than 1. Therefore, one can write:

$$\langle p_0(\tilde{x}_1, \tilde{x}_2)\rangle_t = \mathrm{Re}\left\{\frac{|F_0|^2}{2\rho S c_g}\left[1 + 2\sum_{n=1}^{+\infty}(-1)^n\,\mathrm{e}^{2\mathrm{i}nk_0(\tilde{x}_2-\tilde{x}_1)}\,\mathrm{e}^{-\eta nk_0(\tilde{x}_2-\tilde{x}_1)}\right]\right\}. \tag{24}$$

The evaluation of the expectation with respect to $\tilde{x}_1$ and $\tilde{x}_2$ of the input power is possible. Its expression is:

$$\begin{aligned}\langle\langle p_0(\tilde{x}_1, \tilde{x}_2)\rangle_t\rangle_{x_1,x_2} &= \mathrm{Re}\left\{\frac{|F_0|^2}{2\rho S c_g}\left[1 + 2\sum_{n=1}^{+\infty}(-1)^n\langle\mathrm{e}^{2\mathrm{i}nk_0(\tilde{x}_2-\tilde{x}_1)}\right.\right.\\ &\qquad\left.\left.\times\ \mathrm{e}^{-\eta nk_0(\tilde{x}_2-\tilde{x}_1)}\rangle_{x_1,x_2}\right]\right\}\\ &= \mathrm{Re}\left\{\frac{|F_0|^2}{2\rho S c_g}\left[1 + 2\sum_{n=1}^{+\infty}(-1)^n\,\mathrm{e}^{2\mathrm{i}nk_0(1+\mathrm{i}\eta/2)(x_2-x_1)}\right.\right.\\ &\qquad\left.\left.\times\ \mathrm{e}^{\eta^2n^2k_0^2(\sigma_1^2+\sigma_2^2)/2}\,\mathrm{e}^{-2\mathrm{i}\eta n^2k_0^2(\sigma_1^2+\sigma_2^2)/2}\,\mathrm{e}^{-2n^2k_0^2(\sigma_1^2+\sigma_2^2)}\right]\right\}.\end{aligned} \tag{25}$$

Equation (25) is an expanded form of the expectations with respect to $\tilde{x}_1$ and $\tilde{x}_2$, of the time averaged active power flow. The limit of this series expansion when $k_0\sigma_1$ and $k_0\sigma_2$ grow, is p_{inf}.

The simplifications which have been carried out in the previous developments would still have been valid if another random law had been chosen. For example, one may consider a uniform law which is of major interest to simulate random variables for which the probability of belonging to a specific interval, is proportional to the length of this interval. Its probability density function is $f(y) = 1/(2(a-m))$, where m is the mean and a is the half width of the law. Using the uniform law, one can calculate the expectation of a generic term $\exp[2ik_0x]\exp[2ik_0\varepsilon]$. The choice of this term is relevant when considering for example the expression of the Lagrangian density given by equations (15). ε is a random variable of zero mean. Thus, the expectation is expressed as follows:

$$\langle e^{2ik_0x}\, e^{2ik_0\varepsilon}\rangle_\varepsilon = e^{2ik_0x}\frac{1}{2a}\int_{-a}^{a} e^{2ik_0y}\,dy = e^{2ik_0x}\frac{e^{2ik_0a} - e^{-2ik_0a}}{2ik_0a} = e^{2ik_0x}\frac{\sin(2k_0a)}{k_0a}. \qquad (26)$$

When k_0 is sufficiently high, the expectation of $\exp[2ik_0(x+\varepsilon)]$ converges to zero and consequently it can be shown that the expectation of the Lagrangian density converges to zero when the frequency increases. This particular example could be extended to the different random terms.

Different figures representing the evolution of the time averaged energy density and input power, versus the frequency or the standard deviation, are proposed for the following values of the geometrical and mechanical parameters of the rod: $\rho S = 0{\cdot}78$ kg/m, $ES = 2{\cdot}1 \times 10^7$ N, $L = 20$ m, $x = 10$ m, $\eta = 0{\cdot}02\%$ and $F_0 = 1$ N. The standard deviations of the different random variables are fixed to the same value denoted σ.

Figure 2 is a comparison between the time averaged energy density and the expectations with respect to $\tilde{x}$, $\tilde{x}_1$ and $\tilde{x}_2$, of the time averaged energy density for two values of σ. The expression of the expectation of the energy density is obtained considering equation (15) and replacing A_1 and A_2 by their explicit values, given by equation (22). The calculation of the expectations is then carried out in an analytical way, using the definition (11).

This figure illustrates the growing influence of the random parameters when the frequency increases. Indeed, the low frequency behaviour of the stochastic formulation is in good agreement with the deterministic response. The first eigenfrequencies are still represented. Moreover, the frequency range for which the stochastic formulation and the deterministic calculations provide similar responses, depends on the value of σ. On the other hand, the expectation of the time averaged total energy density gives a smooth trend of the deterministic response in the high frequency domain.

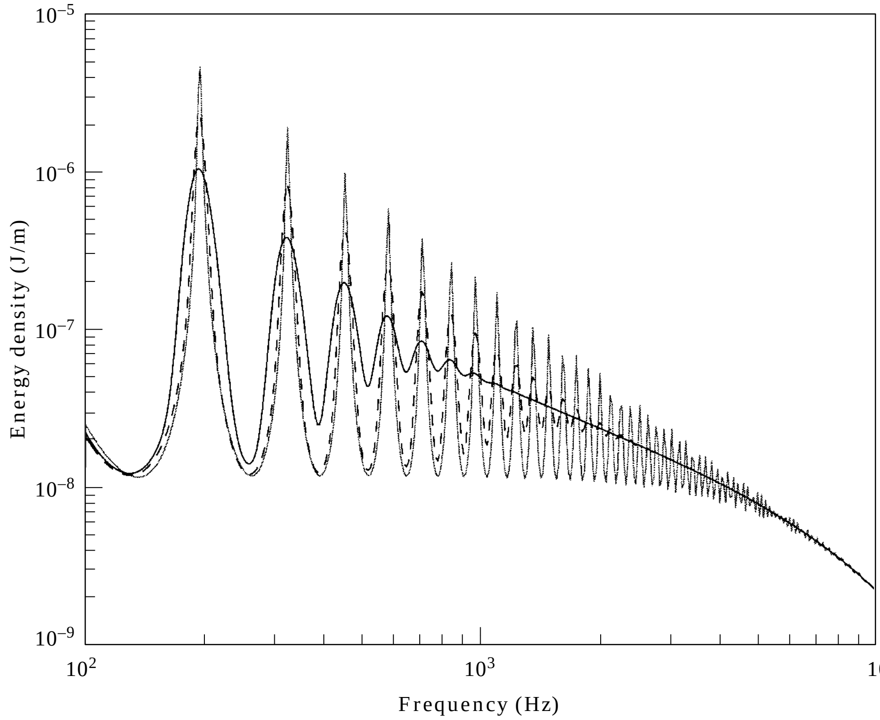

Figure 2. Cantilever rod: a comparison of the frequency evolution of the deterministic energy density (····), and expectations of the energy density, for $\sigma = 0{\cdot}3$ (– – –) and $\sigma = 0{\cdot}8$ (——).

Figure 3 provides a comparison between the energy density obtained by the stochastic formulation and the solution of equation (18) solved for $\langle\langle p_0\rangle_t\rangle_{x_1,x_2} = p_{inf}$. The speed of convergence of the stochastic energy variable to the solution of equation (18) depends on the values of the standard deviations.

Figure 4 illustrates the input power obtained by the stochastic formulation divided by p_{inf}. When the standard deviations are sufficiently high, one can observe that $\langle\langle p_0\rangle_t\rangle_{x_1,x_2}$ is in good agreement with p_{inf} (the ratio converges to 1). Moreover, for high values of the frequency, $\langle\langle p_0\rangle_t\rangle_{x_1,x_2}$ and p_{inf} converge for small values of σ.

The analytical results obtained in this part for the particular case of a cantilever rod, tend to prove that the SEM equations are reobtained by the stochastic formulation as long as $k_0\sigma$ is large enough. Indeed, for a given frequency level, solving equation (18) and using p_{inf} is valid as far as the standard deviation is sufficiently high. If the frequency value is low, the validity domain is obtained for large values of σ. On the other hand, high frequency investigations using the approximate stochastic differential equation (18), is allowed for much smaller values of the standard deviation. Thereby, when referring to the physical definition of the random parameters, one can easily deduce that considering large values of the standard deviation does not make sense. In conclusion, the respect of the physics of the studied structure implies that the simplified energy governing

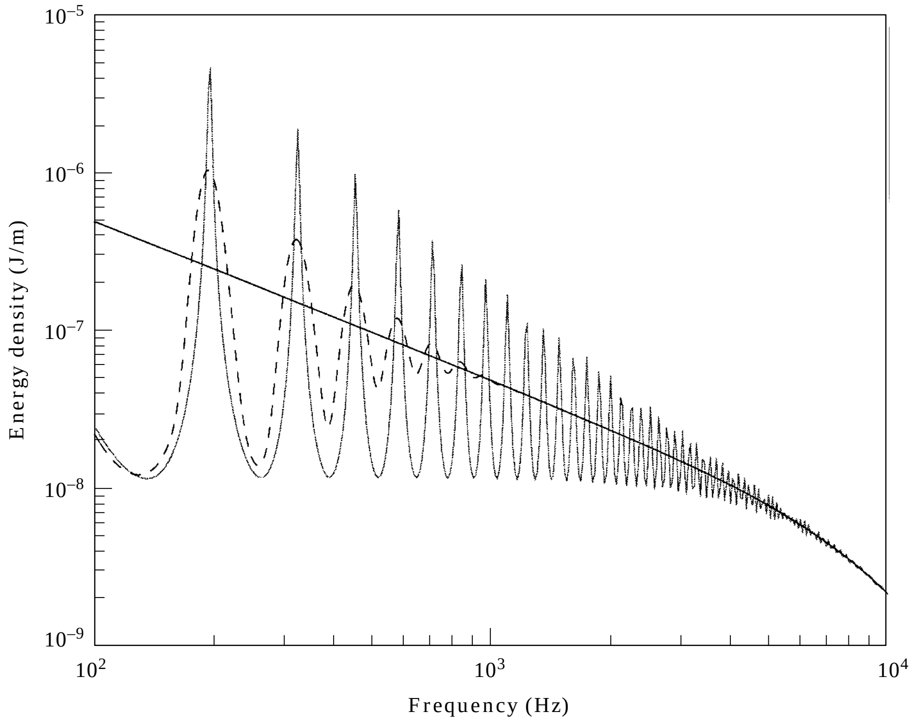

Figure 3. Cantilever rod: frequency evolution of the energy density of the deterministic case (····), solution of equation (17) (——), and stochastic formulation for $\sigma = 0{\cdot}8$ (– – –).

equation as well as p_{inf} must be used only in the high frequency domain. Moreover, the proof concerning the validity of p_{inf} is generalized to any type of rods. Indeed, the expression of p_{inf}, obtained for the particular rod studied in this section, is not

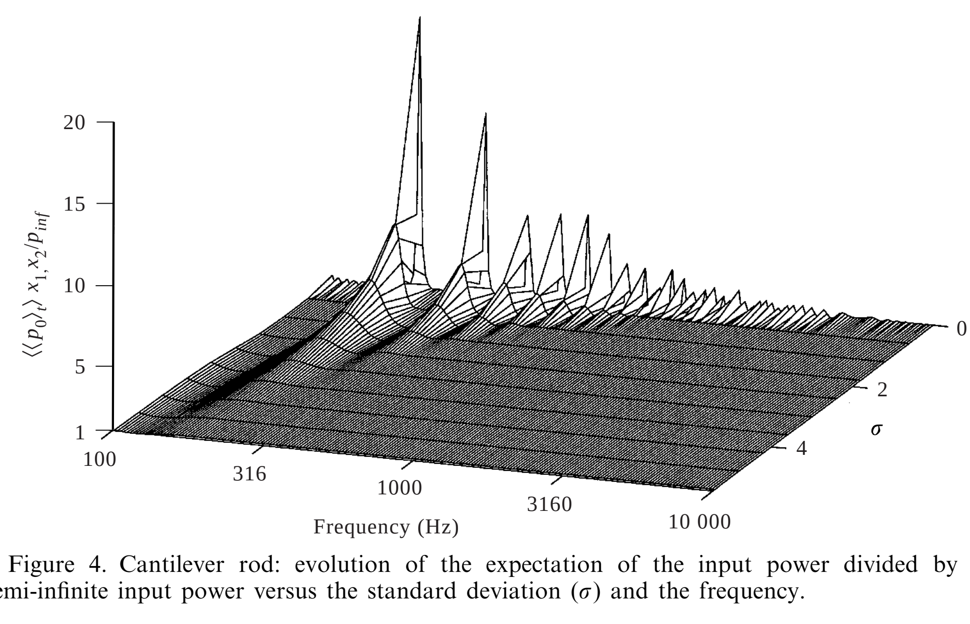

Figure 4. Cantilever rod: evolution of the expectation of the input power divided by the semi-infinite input power versus the standard deviation (σ) and the frequency.

a function of its geometrical description. Therefore, it is assumed that the same expression would have been obtained for any type of rod.

Similar calculations could have been handled if the random parameter had been introduced on another structural parameter. It is assumed that similar conclusions could have been drawn. However, this approach corresponds to a complete reformulation of the problem, taking into account new fundamental random assumptions.

4.3. THE ENERGY RANDOM FORMULATION OF THE BEAM

It will be shown in this section that the main conclusions obtained for the rod remain valid in the case of flexural motion. Thereby, the former process is resumed by expressing the general solution of the governing equation of the beam in terms of the superposition of two propagative waves, and two evanescent waves:

$$\begin{aligned} u(x) = {} & A_1(x_1, x_2)\exp(-\mathrm{i}kx) + A_2(x_1, x_2)\exp(kx) \\ & + B_1(x_1, x_2)\exp(-kx) + B_2(x_1, x_2)\exp(kx), \end{aligned} \tag{27}$$

where u represents the displacement on any subdomain of the beam which does not contain external loadings. The parameter k denotes the complex wave number and $E = E_0(1 + \mathrm{i}\eta)$ is the complex modulus:

$$k \approx k_0(1 - \mathrm{i}\eta/4) = \omega^2 \frac{\rho S}{E_0 I}(1 - \mathrm{i}\eta/4).$$

The parameter I represents the moment of inertia. The expressions of the slope (θ), the bending moment (M), and the shear force (V) are deduced from the expression of the displacement, using the classical relationships [13]:

$$\theta = \frac{\partial u(x)}{\partial x}, \qquad M = -EI\frac{\partial^2 u(x)}{\partial x^2}, \qquad V = EI\frac{\partial^3 u(x)}{\partial x^3}. \tag{28}$$

One can write the analytical expressions of the kinetic and strain energy density and the active power flow as follows:

$$\begin{cases} \langle T(x)\rangle_t = \dfrac{\rho S\omega^2}{4} uu^*, \\[2ex] \langle U(x)\rangle_t = \dfrac{E_0 I}{4}\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial^2 u^*}{\partial x^2}, \\[2ex] \langle I(x)\rangle_t = \mathrm{Re}\left\{\dfrac{-\mathrm{i}\omega}{2}(Vu^* + M\theta^*)\right\}. \end{cases} \tag{29}$$

The contribution of the evanescent wave is neglected far from the loading point and the boundaries. Therefore, the expressions of the time averaged energy density, Lagrangian density and active power flow are only expressed as a function of $A_1(x_1, x_2)$ and $A_2(x_1, x_2)$ as follows (the entire analytical developments may be found in reference [5]):

$$\begin{cases} \langle W(x)\rangle_t = \frac{1}{2}\rho S\omega^2\{|A_1(x_1, x_2)|^2\,\mathrm{e}^{-\eta k_0 x} + |A_2(x_1, x_2)|^2\,\mathrm{e}^{\eta k_0 x} \\ \qquad +\ A_1(x_1, x_2)A_2^*(x_1, x_2)\,\mathrm{e}^{-2\mathrm{i}k_0 x} \\ \qquad +\ A_1^*(x_1, x_2)A_2(x_1, x_2)\,\mathrm{e}^{2\mathrm{i}k_0 x}\} \\ \langle L(x)\rangle_t = 0, \\ \langle I(x)\rangle_t = \frac{-1}{2}\rho S\omega^2 c_g\{|A_1(x_1, x_2)|^2\,\mathrm{e}^{-\eta k_0 x} - |A_2(x_1, x_2)|^2\,\mathrm{e}^{\eta k_0 x}\}. \end{cases} \tag{30}$$

The same operations as for the rod are carried out. First of all, the different geometrical variables (x_1, x_2 and x) are considered as random variables. The notations and definitions introduced in the previous section are utilized in this part with any further explanations.

The expectations with respect to $\tilde{x}$, of the time averaged energy density has the following expression:

$$\begin{aligned} \langle\langle W(\tilde{x})\rangle_t\rangle_x = \tfrac{1}{2}\rho S\omega^2\{&|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2} + |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2} \\ &+\ A_1(\tilde{x}_1, \tilde{x}_2)A_2^*(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{-2\mathrm{i}k_0 x}\,\mathrm{e}^{-2k_0^2\sigma_x^2} \\ &+ A_1^*(\tilde{x}_1, \tilde{x}_2)A_2(\tilde{x}_1, \tilde{x}_2)\,\mathrm{e}^{2\mathrm{i}k_0 x}\,\mathrm{e}^{-2k_0^2\sigma_x^2}\}. \end{aligned} \tag{31}$$

The expectation with respect to $\tilde{x}$ of the active energy flow may be written:

$$\begin{aligned} \langle\langle I(\tilde{x})\rangle_t\rangle_x = \frac{-1}{2}\rho S\omega^2 c_g\{&|A_1(x_1, x_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2} \\ &-\ |A_2(x_1, x_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2}\}. \end{aligned} \tag{32}$$

Equation (31) may be rewritten in a simple way by neglecting the terms multiplied by $\exp[-2k_0^2\sigma_x^2]$. This approximation is valid when the frequency, as well as the standard deviation are sufficiently high. Therefore, the approximate energy variables obtained by the stochastic formulation for the beam may be written:

$$\begin{cases} \langle\langle W(\tilde{x})\rangle_t\rangle_x = \frac{1}{2}\rho S\omega^2\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2} + |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2}\}, \\ \langle\langle I(\tilde{x})\rangle_t\rangle_x = \frac{1}{2}\rho S\omega^2 c_g\{|A_1(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{-\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2} - |A_2(\tilde{x}_1, \tilde{x}_2)|^2\,\mathrm{e}^{\eta k_0 x}\,\mathrm{e}^{\eta^2 k_0^2\sigma_x^2/2}\}. \end{cases} \tag{33}$$

As well as for the rod energy formulation, one can write a relationship valid in the high frequency range, between the expectations of the time averaged active power flow and energy density, expressed by the relations (33). This relationship leads to the standard form of the energy governing equation obtained by Wohlever and Bernhard [4], between the space averages over a half wavelength of the energy density and the injected power. For large values of $k_0\sigma_x$, the stochastic energy governing equation is:

$$-\frac{c_g^2}{\eta\omega}\frac{\partial^2\langle\langle W(\tilde{x})\rangle_t\rangle_x}{\partial x^2}+\eta\omega\langle\langle W(\tilde{x})\rangle_t\rangle_x=0. \quad (34)$$

The expectation with respect to $\tilde{x}_1$ and $\tilde{x}_2$ of equation (34) is considered, in order to replace the input power injected at one boundary of the subdomain on which equation (34) is valid, by the input power of the associated semi-infinite structure. Therefore, x_f is assumed to be equal to x_1 or x_2. One finally obtains the following energy equation:

$$-\frac{c_g^2}{\eta\omega}\frac{\partial^2\langle\langle\langle W(\tilde{x})\rangle_t\rangle_x\rangle_{x_1,x_2}}{\partial x^2}+\eta\omega\langle\langle\langle W(\tilde{x})\rangle_t\rangle_x\rangle_{x_1,x_2}=0. \quad (35)$$

Solving equation (35) requires two boundary conditions corresponding to Neumann, Dirichlet or Cauchy conditions. Moreover, the evaluation of $\langle\langle p_0(\tilde{x}_1,\tilde{x}_2)\rangle_t\rangle_{x_1,x_2}$ is carried out. As well as for the rod, a particular case is considered corresponding to the cantilever beam loaded at $x_f=x_1$ (see Figure 5).

The general analytical expression of $\langle p_0(x_1,x_2)\rangle_t$ is recalled:

$$\langle p_0(x_1,x_2)\rangle_t=-\mathrm{Re}\left\{\frac{\mathrm{i}\omega}{2}F_0(x_1)u^*(x_1)\right\}. \quad (36)$$

A realistic evaluation of the input power must take into account the contribution of the evanescent wave field, whose influence is not negligible near the loading. In the case of a clamped free beam, the displacement at the free end of the beam has the following expression:

$$u(x_1)=\frac{F_0}{k_0^2EI}\frac{\tan k(x_2-x_1)-\tanh k(x_2-x_1)}{1+\dfrac{1}{\cos k(x_2-x_1)\cosh k(x_2-x_1)}}. \quad (37)$$

The expectation of the time averaged power input with respect to $\tilde{x}_1$ and $\tilde{x}_2$ must be developed. In order to simplify the calculation of the expectation of $\langle p_0(x_1,x_2)\rangle_t$, two approximations are proposed: the term $\tanh k(\tilde{x}_2-\tilde{x}_1)$ is considered deterministic; actually, its value converges rapidly to 1 when the frequency increases. The fraction $1/\cos k(x_2-x_1)\cosh k(x_2-x_1)$ is neglected, since the hyperbolic cosine increases quickly.

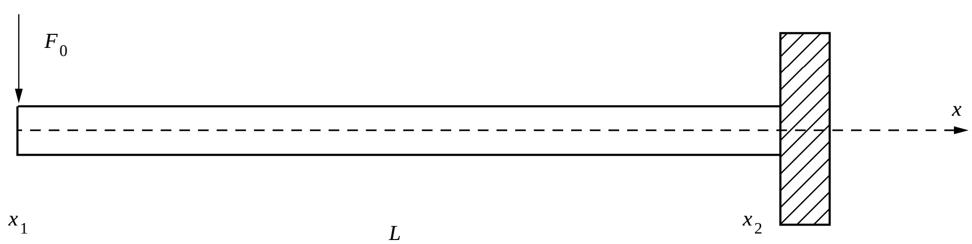

Figure 5. Cantilever beam submitted to a point loading at the free end.

Using the two previous approximations, and expanding the expression of the input power in terms of a Taylor series, one finally obtains:

$$\langle\langle p_0(\tilde{x}_1, \tilde{x}_2)\rangle_t\rangle_{x_1,x_2} \approx \mathrm{Re}\left\{\frac{\mathrm{i}F_0^2\omega}{2k_0^3 EI}\left(1 + \mathrm{i} + 2\mathrm{i}\sum_{n=1}^{+\infty}(-1)^n\,\mathrm{e}^{-2\mathrm{i}nk(x_2 - x_1)}\,\mathrm{e}^{n^2\eta^2k_0^2(\sigma_1^2+\sigma_2^2)/2}\right.\right.$$

$$\left.\left.\times\,\mathrm{e}^{2\mathrm{i}n^2\eta k_0^2(\sigma_1^2+\sigma_2^2)}\,\mathrm{e}^{-2n^2k_0^2(\sigma_1^2+\sigma_2^2)} - \tanh k(x_2 - x_1)\right)\right\}. \qquad (38)$$

A cantilever beam is considered (see Figure 5). Its geometrical and mechanical characteristics are: $\rho S = 0{\cdot}78$ kg/m, $EI = 2{\cdot}1$ N m^2, $L = 2$ m, $\eta = 0{\cdot}02\%$ and $F_0 = 1$ N. The expressions of the input power used in the exact formulation, the SEM, and the stochastic formulation, are numerically evaluated and illustrated versus the frequency (see Figure 6). It is stated that $\sigma_x = \sigma_1 = \sigma_2 = \sigma$.

The ratio of the input power evaluated by the stochastic formulation and the input power of the associated infinite structure are given versus both frequency and standard deviation (see Figure 7). As well as for the case of the rod, these two figures show that the accuracy of the modal description obtained by the stochastic formulation in the low frequency field, depends on the value of the standard deviation.

For a given value of the frequency, Figure 7 illustrates that the expectation of the input power obtained by the stochastic formulation, converges to the input power of the SEM when σ increases.

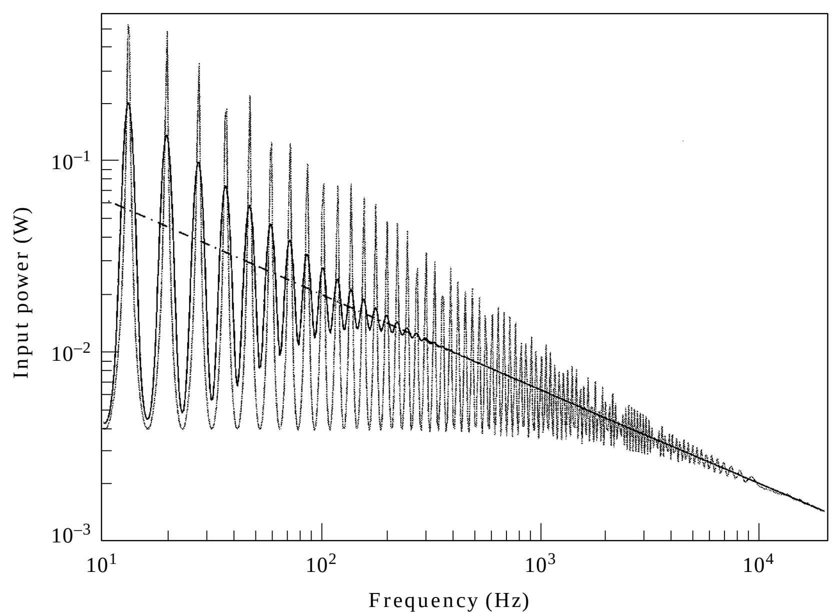

Figure 6. Cantilever beam: a comparison of the frequency evolutions of the deterministic input power (····), its expectation for $\sigma = 0{\cdot}04$ (——) and p_{inf} (–.–).

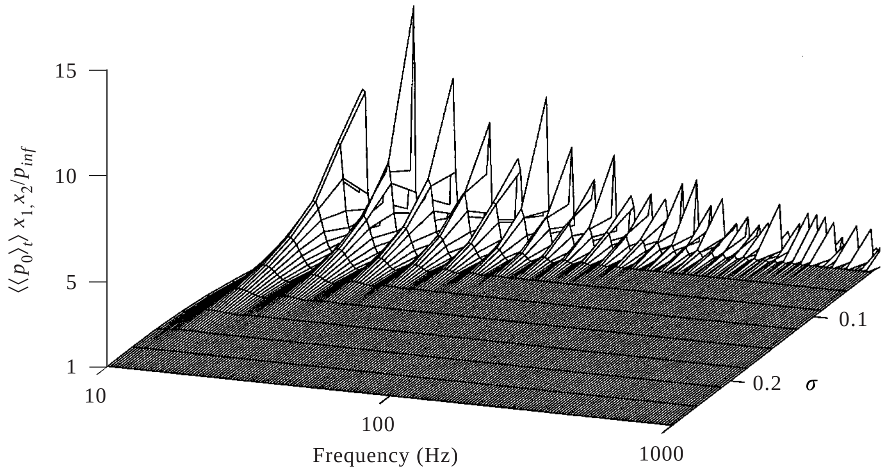

Figure 7. Cantilever beam: evolution of the expectation of the input power divided by p_{inf} versus the standard deviation (σ) and the frequency.

5. CONCLUSION

In this paper, the principal theoretical results obtained by the Simplified Energy Method for isolated bars and beams are recovered, using a random description of the mechanical systems. Indeed, independent Gaussian random variables are introduced in the description of the geometry of the structures: the boundaries, the loading location and the spatial position. In this context, it is proved that for both rods and beams, the expectations of the energy variables with respect to the random variables mentioned previously, satisfy for large values of $k_0\sigma_x$ the standard governing energy equations (18), and (35). Moreover, the stochastic formulation allows the use of p_{inf} when $k_0\sigma_1$ and $k_0\sigma_2$ are sufficiently high. Indeed, the stochastic approach highlights a transition zone connecting the deterministic expression $\langle p_0 \rangle_t$ to p_{inf}.

In other respects, the stochastic formulation is a way to exhibit the bounds of the low, mid and high frequency ranges. Indeed, the low frequency range may be defined as the domain for which the deterministic and the stochastic results give approximately the same results. On the other hand, the high frequency domain is reached when the simplified energy governing equation becomes valid. Obviously, the mid frequency range is the complementary of the two former domains. Thus, the numerical values of the limits of the three domains depend on the values of $k_0\sigma_x$, $k_0\sigma_1$ and $k_0\sigma_2$.

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APPENDIX: EVALUATION OF THE EXPECTATIONS

The procedure for obtaining the different expectations of the term present in the expressions of the energy variables is illustrated. A generic term of the type $\exp[ik(x+\varepsilon)]$ is considered. ε represents a random variable, while $k = k_0(1 - i\eta/2)$ denotes a complex constant. It is assumed that x is deterministic. Consequently, the calculation of the expectation is simply carried out on $\exp[ik\varepsilon]$. ε is a Gaussian random variable of zero mean and standard deviation σ. The probability density function of ε may be written as:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma}\,e^{-y^2/2\sigma^2}. \tag{A1}$$

The expectation of $\exp[ik\varepsilon]$ is then analytically evaluated:

$$\langle e^{ik\varepsilon}\rangle_\varepsilon = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,e^{iky}\,dy. \tag{A2}$$

A change of variable is carried out:

$$X = \frac{y}{\sqrt{2}\sigma}. \tag{A3}$$

Thus, the expectation may be written:

$$\begin{aligned}\langle e^{ik\varepsilon}\rangle_\varepsilon &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-X^2 + ik\sqrt{2}\sigma X}\,dX \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-X^2 + \eta k_0\sqrt{2}/2\sigma X + ik_0\sqrt{2}\sigma X}\,dX \\ &= \frac{e^{\eta^2 k_0^2\sigma^2/8}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(X-\eta k_0\sqrt{2}/4\sigma)^2}\, e^{ik_0\sqrt{2}\sigma X}\,dX. \end{aligned} \tag{A4}$$

A new change of variable is carried out:

$$Y = X - \eta k_0 \frac{\sqrt{2}}{4}\sigma. \tag{A5}$$

Therefore, using the property of unparity of the imaginary part of the function one finally obtains:

$$\begin{aligned}\langle e^{ik\varepsilon}\rangle_\varepsilon &= \frac{e^{\eta^2 k_0^2\sigma^2/8}\, e^{i\eta k_0^2\sigma^2/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-Y^2}\, e^{ik_0\sqrt{2}\sigma Y}\,dY \\ &= \frac{e^{\eta^2 k_0^2\sigma^2/8}\, e^{i\eta k_0^2\sigma^2/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-Y^2}\cos(k_0\sqrt{2}\sigma Y)\,dY. \end{aligned} \tag{A6}$$

The value of the integral is well known and is given in books dealing with classical mathematical functions [14]. The final result is:

$$\begin{aligned}\langle e^{ik\varepsilon}\rangle_\varepsilon &= \frac{e^{\eta^2 k_0^2\sigma^2/8}\, e^{i\eta k_0^2\sigma^2/2}}{\sqrt{\pi}}\sqrt{\pi}\, e^{-k_0^2\sigma^2/2} \\ &= e^{\eta^2 k_0^2\sigma^2/8}\, e^{i\eta k_0^2\sigma^2/2}\, e^{-k_0^2\sigma^2/2}. \end{aligned} \tag{A7}$$

It is also possible to evaluate the expectations of more simple terms appearing in the energy expressions, such as $\exp[ik_0\varepsilon]$ and $\exp[\eta k_0\varepsilon/2]$. One finally obtains:

$$\begin{aligned}\langle e^{ik_0\varepsilon}\rangle_\varepsilon &= e^{-k_0^2\sigma^2/2}, \\ \langle e^{\eta k_0\varepsilon/2}\rangle_\varepsilon &= e^{\eta^2 k_0^2\sigma^2/8}. \end{aligned} \tag{A8}$$